

Tales of Hoffman

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Abstract

Hoffman's bound on the chromatic number of a graph states that $\chi \geq 1 - \frac{\lambda_1}{\lambda_n}$. Here we show that the same bound, or slight modifications of it, hold for several graph parameters related to the chromatic number: the vector coloring number, the ψ -covering number and the λ -clustering number.

1 Introduction

Hoffman's bound: Let G be a graph on n vertices, χ its chromatic number, and A its adjacency matrix. Let λ_1 and λ_n the largest and least eigenvalues of A . A theorem of Hoffman [2] states that:

$$\chi \geq 1 - \frac{\lambda_1}{\lambda_n}.$$

The vector chromatic number: Karger, Motwani and Sudan [3] define a quadratic programming relaxation of the chromatic number, called the *vector chromatic number*. This is the minimal k such that there exist unit vectors $u_1, \dots, u_n \in \mathbb{R}^n$ with:

$$\langle u_i, u_j \rangle \leq -\frac{1}{k-1},$$

whenever (i, j) is an edge in the graph.

Let χ_v denote the vector chromatic number of G . Karger, Motwani and Sudan observe that $\chi_v \leq \chi$. In this note we show that Hoffman's bound holds for this parameter as well:

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Theorem 1 *Let G be a graph. Let $W \neq 0$ be a symmetric matrix such that $W_{i,j} = 0$ whenever $(i, j) \notin E$. Let λ_1 and λ_n be the largest and least eigenvalues of W . Then*

$$\chi_v(G) \geq 1 - \frac{\lambda_1}{\lambda_n}.$$

The ψ -covering number: Let ψ be a graph parameter, such that $\psi = 1$ on graphs with no edges, and for every graph G , $\psi(G) \geq \chi(G)$. The ψ -covering number of a graph G was defined by Amit, Linial and Matoušek [1] to be the minimal k such that there exist k subsets of G , S_1, \dots, S_k so that for every $v \in G$, $\sum_{i: v \in S_i} \frac{1}{\psi(G[S_i])} \geq 1$. They show that this value is bounded between $\sqrt{\chi(G)}$ and $\chi(G)$, and ask whether better lower bounds can be proven when $\psi(G) = \text{dgn}(G) + 1$ (the degeneracy of $G + 1$), and $\psi(G) = \Delta(G) + 1$ (the maximal degree in $G + 1$). To state our result, we'll need a couple of ad-hoc definitions:

Definition 1 *A graph G has c -vertex cover if there exists a cover $E(G) = \cup_{i \in V(G)} E_i$ such that for all $i \in V(G)$, $E_i \subset \{e \in E : i \in e\}$, and $|E_i| \leq c$.*

Denote by $L_{\psi, \alpha}(G)$ the minimal k such that there exist k subsets of G , S_1, \dots, S_k , so that for all $i = 1, \dots, k$, $G[S_i]$ has a $\alpha \cdot \psi(G)$ -vertex cover, and for every $v \in G$, $\sum_{i: v \in S_i} \frac{1}{\psi(G[S_i])} \geq 1$.

Observe that all graphs have a $\psi(G)$ -vertex cover for $\psi(G) = \text{dgn}(G) + 1$, and a $\frac{1}{2}\psi(G)$ -vertex cover when $\psi(G) = \Delta(G) + 1$. So in these cases, $L_{\psi, 1}$ and $L_{\psi, \frac{1}{2}}$, respectively, are exactly the ψ -covering numbers.

Note also that $\alpha = 0$ means that the S_i are independent sets. Thus, since $\psi = 1$ on such sets, $L_{\psi, 0} = \chi$.

Theorem 2

$$L_{\psi, \alpha}(G) \geq \frac{d - \lambda_n}{2\alpha - \lambda_n},$$

where λ_n is the least eigenvalue of G , and d the average degree.

Note that when the graph is regular and $\alpha = 0$, this is the same as Hoffman's bound. For random d -regular graphs, $|\lambda_n| = O(\sqrt{d})$ and $\chi = \Theta(\frac{d}{\log d})$. So in this case (if α is taken small) the bound is slightly better than $\sqrt{\chi}$ mentioned above.

The λ -clustering number: Finally, we are interested in a graph parameter that has to do with how well a graph can be partitioned into sparse clusters:

Definition 2 *Let W be a weighted adjacency matrix of a graph G . A partition $V = \dot{\cup}_{i=1}^k C_i$ is a λ -clustering of G into k clusters if*

$$\max_{i \in [k]} \lambda_1(C_i) \leq \lambda,$$

where $\lambda_1(C_i)$ is the largest eigenvalue of the (weighted) sub-graph spanned by the vertices in C_i .

The λ -clustering number of G is the minimal k such that there exists a λ -clustering of G into k clusters.

It is not hard to see that the 0-clustering number is identical to the chromatic number.

We show that Hoffman's bound can also be extended to this graph parameter:

Theorem 3 *Let W be a weighted adjacency matrix. Let λ_1 and λ_n the largest and least eigenvalues of W . The λ -clustering number of the graph is at least:*

$$\frac{\lambda_1 - \lambda_n}{\lambda - \lambda_n}.$$

2 Vectorial characterization of the least eigenvalue

All three results mentioned in the previous section rely on the following observation:

Lemma 4 *Let A be a real symmetric matrix and λ_n its least eigenvalue.*

$$\lambda_n = \min \frac{\sum_{i,j=1}^n A_{i,j} \langle v_i, v_j \rangle}{\sum_{i=1}^n \|v_i\|_2^2}. \quad (1)$$

where the minimum is taken over all $v_1, \dots, v_n \in \mathbb{R}^n$.

Proof: By the Rayleigh-Ritz characterization, λ_n equals

$$\begin{aligned} & \min \sum_{i,j} A_{i,j} x_i x_j \\ & \text{s.t. } x \in \mathbb{R}^n \\ & \|x\|_2 = 1. \end{aligned}$$

Denote by PSD_n the cone of $n \times n$ positive semi-definite matrices. For each unit vector $x \in \mathbb{R}^n$, let X be the matrix $X_{i,j} = x_i x_j$. This is a positive semi-definite matrix of rank 1 and trace 1, and all such matrices are obtained in this way. Hence, λ_n equals:

$$\begin{aligned} & \min \sum_{i,j} A_{i,j} X_{i,j} \\ & \text{s.t. } X \in PSD_n \\ & \text{rank}(X) = 1 \\ & \text{tr}(X) = 1. \end{aligned}$$

However, the rank restriction is superfluous. It restricts the solution to an extreme ray of the cone PSD_n , but, by convexity, the optimum is attained on an extreme ray anyway. Hence, λ_n equals:

$$\begin{aligned} \min \quad & \sum_{i,j} A_{i,j} X_{i,j} \\ \text{s.t.} \quad & X \in PSD_n \\ & \text{tr}(X) = 1. \end{aligned}$$

Now, think of each $X \in PSD_n$ as a Gram matrix of n vectors, v_1, \dots, v_n (i.e. $X_{i,j} = \langle v_i, v_j \rangle$). An equivalent formulation of the above is thus:

$$\begin{aligned} \min \quad & \sum_{i,j} A_{i,j} \langle v_i, v_j \rangle \\ \text{s.t.} \quad & v_i \in \mathbb{R}^n \quad \text{for } i = 1, \dots, n \\ & \sum_{i=1}^n \|v_i\|_2^2 = 1. \end{aligned}$$

Clearly, this is equivalent to 1. \blacksquare

3 Proofs of the theorems

Proof (Theorem 1): Let G be a graph on n vertices with vector chromatic number χ_v . Let $W \neq 0$ be a symmetric matrix such that $W_{i,j} = 0$ whenever $(i, j) \notin E$. Let λ_1 and λ_n be the largest and least eigenvalues of W .

We choose vectors v_1, \dots, v_n , and look at the bound they give on λ_n in Lemma 4. Let $u_1, \dots, u_n \in S^n$ be vectors on which the vector chromatic number is attained. That is, $\langle u_i, u_j \rangle \leq -\frac{1}{\chi_v - 1}$ for $(i, j) \in E$, and $\|u_i\|_2 = 1$. Let $\alpha \in \mathbb{R}^n$ be an eigenvector of W corresponding to λ_1 . Set $v_i = \alpha_i \cdot u_i$.

Since $W_{i,j} = 0$ whenever $\langle u_i, u_j \rangle > -\frac{1}{\chi_v - 1}$, by Lemma 4,

$$\begin{aligned} \lambda_n &\leq \frac{\sum_{i,j} W_{i,j} \alpha_i \alpha_j \langle u_i, u_j \rangle}{\sum_{i=1}^n \alpha_i^2 \cdot \|u_i\|_2^2} \leq -\frac{1}{\chi_v - 1} \cdot \frac{\sum_{i,j} W_{i,j} \alpha_i \alpha_j}{\sum_i \alpha_i^2} = \\ &= -\frac{1}{\chi_v - 1} \cdot \frac{\alpha^t W \alpha}{\|\alpha\|^2} = -\frac{1}{\chi_v - 1} \cdot \lambda_1. \end{aligned}$$

Equivalently, $\chi_v \geq 1 - \frac{\lambda_1}{\lambda_n}$, as claimed. \blacksquare

Proof (Theorem 2): Denote $k = L_{\psi, \alpha}(G)$, and let u_1, \dots, u_k be the vertices of the regular $(k-1)$ -dimensional simplex centered at 0 - i.e. $\langle u_i, u_j \rangle = 1$ when $i = j$ and $\frac{-1}{k-1}$ otherwise. Again we choose vectors v_1, \dots, v_n . We do so probabilistically. Let S_1, \dots, S_k be the subsets attaining the value k . For each i , v_i will be chosen from among the u_j 's such that $i \in S_j$. Specifically, let $h_i = \sum_{j: i \in S_j} \frac{1}{\psi(S_j)}$. The probability that v_i is chosen to be u_j is $p_{i,j} = h_i^{-1} \frac{1}{\psi(S_j)}$. Note that $h_i \geq 1$, and so $p_{i,j} \leq \frac{1}{\psi(S_j)}$. (there is a slight abuse of notation here - by $\psi(S_j)$ we refer to $\psi(G[S_j])$.)

Say that an edge is “bad” if both its endpoints are assigned the same vector. For a given j , the probability that an edge $(i, i') \in E(S_j)$ is “bad” because both endpoints were assigned to u_j is $p_{i,j} p_{i',j}$. Thus, the expected number of “bad” edges is at most:

$$\sum_{j=1}^k \sum_{(i,i') \in E(S_j)} p_{i,j} p_{i',j}.$$

Each S_j has a $\alpha \cdot \psi(G)$ -vertex cover $E(S_j) = \cup E_j^i$. Summing the expression above according to this cover (some edges might be counted more than once) we get that the expected number of “bad” edges is at most:

$$\begin{aligned} \sum_{j=1}^k \sum_{i \in S_j} \sum_{i' : (i,i') \in E_j^i} p_{i,j} p_{i',j} &\leq \sum_{j=1}^k \sum_{i \in S_j} p_{i,j} |E_j^i| \frac{1}{\psi(S_j)} \leq \sum_{j=1}^k \sum_{i \in S_j} p_{i,j} \alpha \\ &= \alpha \sum_{i \in G} \sum_{j: i \in S_j} p_{i,j} = \alpha n. \end{aligned}$$

In particular, there is a choice of v_i 's such that the number of “bad” edges is at most αn . Assume this is the case. If $(i, j) \in E(G)$ is a “bad” edge then $\langle v_i, v_j \rangle = 1$. Otherwise $\langle v_i, v_j \rangle = -\frac{1}{k-1}$.

Lemma 4 now gives:

$$\lambda_n \leq \left(2\alpha n - \frac{1}{k-1}(dn - 2\alpha n) \right) / n = \frac{2k\alpha}{k-1} - \frac{d}{k-1},$$

or $(k-1)\lambda_n \leq 2k\alpha - d$. Equivalently, $k \geq \frac{d-\lambda_n}{2\alpha-\lambda_n}$. ■

Note 5 In the definition of the ψ -covering number, and of $L_{\psi, \alpha}$, it is required that for every $v \in G$, $\sum_{i: v \in S_i} \frac{1}{\psi(S_i)} \geq 1$. The bounds given in [1] hold also if we demand that all sums equal 1. In this case, the theorem holds also if we relax the condition that all S_i have an $\alpha \cdot \psi(G)$ -vertex cover, and require only that for each S_i , $|E(S_i)| \leq \alpha \cdot \psi(S_i) \cdot |V(S_i)|$.

Proof (Theorem 3): Denote the λ -clustering number of W by k . Let $u_1, \dots, u_k \in \mathbb{R}^n$ be the vertices of a regular simplex centered at the origin, as above. Let $\alpha \in \mathbb{R}^n$ be an eigenvector of W , corresponding to λ_1 . Let C_1, \dots, C_k be a λ -clustering of G . Define $\phi : V \rightarrow [k]$ to be the index of the cluster containing a vertex. That is, $i \in C_{\phi(i)}$. Define W_t to be the weighted adjacency matrix of the sub-graph spanned by C_t (So $\lambda_1(W_t) \leq \lambda$). Set $v_i = \alpha_i \cdot u_{\phi(i)}$. By Lemma 4,

$$\begin{aligned}
\lambda_n &\leq \frac{\sum_{i,j} \alpha_i \alpha_j \langle u_{\phi(j)}, u_{\phi(j)} \rangle W_{i,j}}{\sum_{i=1}^n \alpha_i^2 \cdot \|u_{\phi(i)}\|_2^2} \\
&= -\frac{1}{k-1} \cdot \frac{\sum_{i,j: \phi(i) \neq \phi(j)} \alpha_i \alpha_j W_{i,j}}{\sum_i \alpha_i^2} + \frac{\sum_{i,j: \phi(i) = \phi(j)} \alpha_i \alpha_j W_{i,j}}{\sum_i \alpha_i^2} \\
&= -\frac{1}{k-1} \cdot \frac{\alpha^t W \alpha}{\|\alpha\|^2} + \frac{k}{k-1} \frac{\sum_{t=1}^k \sum_{i,j \in C_t} \alpha_i \alpha_j W_{i,j}}{\sum_i \alpha_i^2} \\
&\leq -\frac{1}{k-1} \lambda_1 + \frac{k}{k-1} \lambda
\end{aligned}$$

Equivalently, $k \geq \frac{\lambda_n - \lambda_1}{\lambda_n - \lambda}$ ■

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References

- [1] A. Amit, N. Linial, and J. Matoušek. Random lifts of graphs: independence and chromatic number. *Random Structures Algorithms*, 20(1):1–22, 2002.
- [2] A. J. Hoffman. On eigenvalues and colorings of graphs. In *Graph Theory and its Applications (Proc. Advanced Sem., Math. Research Center, Univ. of Wisconsin, Madison, Wis., 1969)*, pages 79–91. Academic Press, New York, 1970.
- [3] D. Karger, R. Motwani, and M. Sudan. Approximate graph coloring by semidefinite programming. *J. ACM*, 45(2):246–265, 1998.